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Continuous-time production, distribution and financial planning with periodic liquidity balancing

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Abstract Due to the inevitable focus on core competencies, even small- and medium-sized companies are increasingly forced to form supply chain (SC) networks. However, their specific situation is often characterized by a lack of equity and limited access to capital markets, so that bank loans must then be used to initiate production and distribution. Within a short-term multi-day planning horizon, both operations and finance must be scheduled precisely in order to obtain practical instructions for each network partner and the network managers. For this purpose, continuous-time modeling is required. Additionally, a coordination of monetary consequences resulting from both site-specific operational events and network-wide financial transactions is necessary to prevent insolvency. As bank overdrafts can be used to overcome financial imbalances during short periods (e.g., days or even hours), appropriate time intervals for liquidity management should be determined. The implementation of these intervals requires discrete-time modeling. In this context, the main challenge is to combine both of the aforementioned modeling techniques within a common decision model. To address this problem, a novel mixed-integer nonlinear program (MINLP) is developed, which enables exact planning and scheduling of SC operations as well as related financial transactions on the one hand, and periodic liquidity balancing on the other hand. A numerical analysis was based on a test scenario with randomly generated data. As we found out that even small problem instances of the MINLP, e.g., a three-stage supply chain with three sites in each stage, were not computable with high-performance hardware and a commercial nonlinear standard solver, we additionally propose an equivalent linearized version of the decision model. The latter could be optimized within acceptable computation time using the CPLEX solver.

Keywords Supply chain networks · Short-term planning and scheduling · Continuous-time modeling · Financing · Liquidity balancing

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1 Introduction

To withstand increasing competitive pressure in global markets, most companies are forced to participate in supply chain (SC) networks. Often, these are companies with a small capital base, but with promising production know-how and market opportunities. Financing is required to bridge the interval between operations and sales. Limited access to capital markets may result in the use of short-term credits or even bank overdrafts.

In the following analysis, a multistage and multi-product SC is considered within a planning horizon of several days. In this context, the planning of operations and financing necessitates continuous-time modeling of production, transportation, sales and financial transactions, in order to determine the starting and ending times exactly to the minute. However, liquidity management must accompany this planning. The liquidity of an organizational structure, such as a SC network, entails its ability to make payments as they fall due. It should be managed within short-term intervals (e.g., day-to-day basis), in order to prevent insolvency (Moir 1997, pp. 1–5). For this purpose, all monetary consequences which can be assigned to such an interval must be balanced, taking profit maximization at the end of the planning horizon into account. Optimizing the problem described above requires an innovative coordination of short-term SC planning and financial planning, based on a combination of continuous-time and discrete-time planning.

The article is structured as follows. Section 2 contains a literature review of other relevant contributions, revealing that the presented optimization model offers an appropriate conceptual approach to solving the abovementioned problem and extends existing research in the field. The mathematical formulation of the mixed-integer nonlinear program (MINLP) is presented in Sect. 3. An equivalent linearization of the MINLP is carried out in Sect. 4. In this context, the model is transformed into a mixed-integer linear program (MILP) in order to improve the computability. A numerical analysis with 30 scenarios composed of randomly generated data can be found in Sect. 5.

2 Literature review

The planning of SC operations is based on the integration of production, distribution and transportation (Erengüç et al. 1999). Besides static formulations, such as the production–distribution coordination model for multistage networks with multiple products by Jayaraman and Pirkul (2001), that do not consider a subdivision of the planning horizon, the overwhelming majority of approaches is based on discrete-time modeling with time periods of fixed length represented by a specific index. In contrast, continuous-time approaches allow for scheduling of production and financings by determining start and end times exactly.

2.1 Discrete-time modeling

Lee and Kim (2002) consider integrated production and distribution planning with capacity constraints in SCs. They propose a multi-period, multi-product, multi-shop model that minimizes the overall costs of production, distribution, inventory holding and shortage costs. The demand of retailers is to be satisfied in given time periods. The authors are aware of the dynamic problem of time consumption, but they only include operation and distribution times that cannot exceed given limits within time periods of fixed length. Park (2005) proposes a mixed-integer approach that integrates production and distribution planning in SC management. The objective is to maximize the total net profit over the time periods the planning horizon is divided into. He distinguishes core business demand (needs to be satisfied necessarily) and forecasted demand (considered as stock-out if it remains unsatisfied). Again, the usage of time-related parameters (e.g., unit processing times, setup times) is limited to ensure the compliance of processes with the available production capacity in the time periods. In general, even an additional integration of periodic financial transactions is known in the literature. For this purpose, Guillén-Gosálbez et al. (2006) propose a MILP for simultaneous operative and financial decisions regarding the management of chemical SCs. They consider a multi-product, multi-echelon distribution network with multipurpose batch plants. An integrated planning and budgeting model is used to generate operative plans. The latter are based on discrete-time scheduling with a rolling horizon and include production planning and cash balancing for each time period. For financing, marketable securities and short-term financing sources are available. Laínez et al. (2007) combine process operations and finances taking strategic decisions of facility opening and capacity increment into account. Their objective is to maximize the corporate value determined at the end of the planning horizon. For periodic liquidity control within their discrete-time model, they define cash at each period as a function of different components (cash at the previous period, the exogenous cash from the sales of products, from any other inflow of cash, the amount borrowed or repaid to the short-term credit line and the raw materials, production and transport payments). Hahn and Kuhn (2011) combine the physical and the financial domain of SC management. They consider a make-to-stock SC that covers procurement, production, distribution and sales at different sites of a network. With regard to their value-based MILP approach, they assume a mid-term planning horizon that is split up into time periods of one month each. For these periods, cash flows of operations (e.g., production, transportation, storage) are merged. One-period borrowings and investments are used for financial management. However, as mentioned by the authors themselves, detailed decisions on quantities and financial positions would be part of short-term approaches that could be part of a hierarchical planning framework. Martins and Quelhas (2016) propose mixed-integer discrete-time approaches that combine financial, production and workforce planning of a company manufacturing a single product or a set of homogenous products. They consider two time horizons both running over the same stream of time periods with fixed length. Whereas bank loans with periodic amortizations are relevant with regard to the long-term level, cash balancing for each of the time periods is considered with regard to the short-term level. As the authors focus on financing labor cost, capital loans with short-term maturity can be used to provide an adequate workforce load for production. Related cash flows (including amortizations and interests) are merged with those that result from costs for production and salaries as well as sales net profits. As monetary consequences are assigned to the beginning or the end of the time periods, the problem ignores issues of timing within a time period. In summary, the aforementioned issue is a general problem of all discrete-time approaches. Although integrating different aspects of SC management (and even the financing of related processes), time discretization inherently implies that the starting and ending times of operational and financial events cannot be determined exactly.

2.2 Continuous-time modeling

In contrast, even a number of continuous-time approaches dealing with scheduling exists in literature; remaining on the level of physical operations, they can be classified by the types of processes to be scheduled and the number of facilities to be captured by scheduling.

I. First, there is intra-organizational modeling focusing on production scheduling within single facilities. Mockus and Reklaitis (1999) propose an approach for plants that combine given batch tasks (with given processing times) and continuous tasks (with processing rates within certain ranges) that need to be scheduled and sequenced. In the resulting state-task-network, they consider profit maximization without addressing the financing of processes. Their continuous-time representation is characterized by a division of the planning horizon into a number of intervals of unequal duration, while processing times are equal to the differences in output times. Although each demand has to be satisfied in full, variable and fixed penalties for missed demand are considered. Floudas and Lin (2004) reveal the benefits of continuous-time in comparison to discrete-time modeling for processes in chemical companies. Besides the aforementioned sequential processes, they also consider continuous-time scheduling of "general network-represented processes" that allow batches to merge and split, and thus require an explicit focus on mass balancing. If continuous variables are to be introduced in order to determine the exact timings of events, the resulting "global event based models" require binary variables that are used to assign state changes of the system (e.g., the start or end of a process) to the events. However, this may not only be accompanied by problems of linearization, but also by problems of estimation (number of events or time points). Mohammadi et al. (2012) consider a job shop problem in order to optimize process plans consisting of several operations needed to manufacture specific parts. The continuous-time approach with multiple objectives (minimizing preparation times or total tardiness) is slot-based, i.e., the authors postulate asynchronous time slots for each machine the products can be assigned to. The length of the time slots is variable, and thus, to be determined by the model. With respect to sequence-dependent preparation times, the start and end times of operations that are assigned to these time slots must be scheduled. Focusing the combination of sequencing and scheduling, the model does not consider costs or profits. Mokhtari et al. (2012) describe a mixed-integer nonlinear flow shop model with outsourcing decisions. The problem is to assign operations (required to fulfill orders) to appropriate resources (i.e., inside or outside machines) so as to minimize the costs of outsourcing (processing and transportation) and costs of mean weighted flow time simultaneously. There are given processing times of the own manufacturer and the subcontractors. The start time of the operations is to be determined. With respect to a given number of orders and limited in-house capacities, they enable outsourcing instead of including profit-based decisions on the overall demand satisfaction. Gomes et al. (2013) consider flexible job shop scheduling in the make-to-order industry. The modeling is adjusted to the integration of jobs that necessitate re-circulation (i.e., jobs that can visit machine groups more than once). Thus, the start time of components on machine groups are to be determined for different stages that represent multiple visits. As the components need to be assembled to form an order, the determination of the start times of order assembly is additionally required. The objective is to minimize the weighted sum of order earliness, order tardiness and intermediate storage times. Baumann and Trautmann (2013) model short-term scheduling within a make-and-pack plant that combines production processes (operations on parallel, nonidentical groups of machines), storage processes (storage tanks with limited capacities) and packing processes (nonidentical parallel packing lines with continuous material flow). The objective is to minimize the makespan of the overall production schedule, while the continuous start times of the required tasks are determined. The due date for the packed products is the end of the planning horizon. Günther (2014) proposes an MILP approach to lot sizing and scheduling, which aims at a minimization of makespan. The modeling is based on the block planning principle, i.e., several product types are integrated into a product family that is scheduled block-wise in a predetermined sequence. Both the start time and the duration of the blocks are to be optimized within given boundaries. The author assumes given demand that needs to be satisfied in full. Focusing on the meeting of given deadlines within single production facilities, time-based objectives are suitable in case of the latter three models.

II. There are approaches that expand scheduling of processes within single production facilities by the integration of distribution decisions. Ullrich (2013) models production and distribution scheduling within two

MILP subproblems. In the first subproblem, he schedules jobs on identical parallel machines. With respect to job-dependent processing times and due dates, the continuous completion times of the jobs need to be determined. The latter are parameters in the second subproblem (vehicle routing problem), if the successive approach is considered. Here, continuous delivery times of the jobs are to be calculated in coordination with the start times of the tours. There are earliest and latest delivery dates that form the bounds of delivery time windows. The objective is to minimize total tardiness. Chang et al. (2013) mention the increasing need for integrated scheduling of production and distribution due to reductions in inventory levels and short lead times. They present an integrated two-level scheduling MINLP. Again, the production level is modeled by an identical parallel machine problem, and the distribution level by a vehicle routing problem. Completion times and delivery times of the jobs are to be determined, while the weighted sum of the total weighted job completion time and total distribution costs is minimized. Low et al. (2013) propose a MINLP for integrated scheduling that coordinates the processing of orders in a distribution center and the delivery to retailers. The latter are characterized by given demand, which is to be satisfied in full within given time windows. Delivery is organized in tours of vehicles that start at the distribution center. As production and distribution can occur continuously throughout the planning horizon, the makespans and the departure times at the retailers are continuous decision variables of the model formulation. The objective is to minimize the total completion time. Despite the well-coordinated integration of production and distribution in the three latter continuous-time model formulations, they are not applicable to our problem as they focus on vehicle routing. Furthermore, there was no need to coordinate distribution with more than one production facility in case of the aforementioned approaches.

III. Finally, there are continuous-time approaches that consider modeling of SC networks, which consist of a definable number of stages, whereas each stage can comprise several sites performing the same or at least homogenous processes. In this context, Steinrücke (2011) expands the focus of continuous-time scheduling to integration of production and distribution scheduling within inter-organizational networks. In this context, he develops a combined planning and scheduling model for SCs that considers the coordination of production, distribution, transportation and scheduling of quantities at sites of different SC stages. This allows for planning material flows exactly to the minute within a planning horizon of several days, while each site can supply every site in the succeeding production stage and can be supplied by any site from the preceding production stage. Regarding possible temporary storages before and/or after site production, Steinrücke (2015) proposes a generalized planning and scheduling approach, which enables that either stock-free material flows within the entire network or site-specific storage times could be prescribed. Both the aforementioned papers contain delivery deadlines. A monetary objective that minimizes the sum of production and transportation costs as well as bonus payments, granted by the customer for early deliveries, is chosen. Thus, scheduling of sales (which is limited to a single final customer, as both the approaches do not include a market stage with different market sites in it) is only possible with regard to the given due date, neglecting the possible existence of time windows. Focusing on costs and not on profits, it is not possible to decide on partial fulfillment of orders. Furthermore, there is a fixed sequence of SC stages that have to be passed through by the material flows. Hence, the possibility of selling intermediate products to the final customer cannot be considered.

2.3 Combined continuous-time and discrete-time modeling

With regard to our problem, which requires the coordination of continuous-time scheduling of events with discrete-time periods of fixed length, it can be stated that there are even few approaches in literature that combine both forms of time representation in a different context. Li and Ierapetritou (2009) propose a successive bi-level optimization problem that is applicable in multi-purpose multi-product batch plants. It consists of a planning problem (upper level) that minimizes the sum of inventory costs, backorder costs and production costs that can occur in different time periods of the planning horizon. The production costs are determined through subproblems (lower level) that schedule different tasks by determining their start time and end time within the time periods. In order to overcome the disadvantages of successive planning, Shah and Ierapetritou (2012) model a SC network consisting of multiple batch production sites and multiple markets simultaneously. Although their planning horizon is discretized into time periods of fixed length (daily production periods), their approach inter-connects discrete-time decisions of the planning level with continuous-time decisions of the scheduling level by production and inventory constraints. On the planning level, production, inventory and shipping targets are predicted for each product. After the assignment of the production targets to the sites, the start times and end times of the tasks that are to be performed at the sites within time periods are determined. Their model formulation based on cost minimization allows for the demand (which is a priori assigned to specific time periods) to be carried over to the next planning period via backordering (which is penalized by specific costs). Hence, the model does not include decisions on the time point of sale on the one hand, and profit-based decisions on the overall demand satisfaction on the other hand. Although both the aforementioned papers coordinate continuous-time scheduling with discrete-time planning periods, they are characterized by the usage of pre-defined event points that represent possible beginnings of processes. In contrast, our model manages without event points, as the number of events to be scheduled is determined within the optimization.

2.4 Contributions in light of the literature review

In summary, the integration of production, distribution and transportation is a well-researched area in SC management. Models with limitations regarding the number of SC stages and sites can be regarded as special cases of general network formulations. There are a number of approaches that are based on the continuous-time representation. The latter is essential for short-term modeling as it allows for the starting and ending times of events to be determined exactly to the minute. Although even the consideration of financial transactions in SC management can be found in publications of recent years, none of them contains continuous-time approaches, to the best of our knowledge.

Besides specific details of our network structure, the main contributions of this paper are:

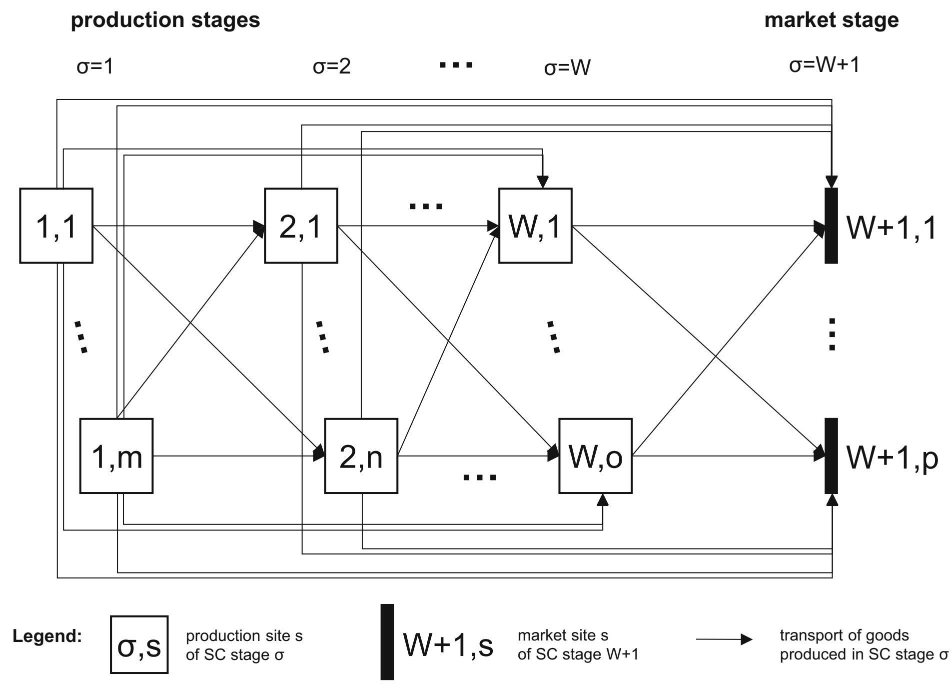
Our model integrates aspects of financing in SC planning and scheduling, which is based on a short-term horizon with a continuous-time axis. It allows for the determination of starting times and amounts of available financings that are required to bridge the interval between operations and sales. As these financings need to be coordinated with the monetary consequences of other simultaneously scheduled events (i.e., production, transportation, sales) in time periods that are relevant for preventing insolvency, a specific liquidity-related connection of continuous-time and discrete-time modeling is established.

3 Conceptual approach and background

The approach to be described in the following is applicable to networks consisting of production sites that allow for continuous production, i.e., 24/7 operations are possible. The latter is typical for the chemical industry or the pharmaceutical industry. All the network sites are legally independent. Thus, they can be additionally contracted for production that is not associated with the considered network. The latter is possible before and after the starting times and ending times of operations, respectively, which need to be determined by the following model. The specifications of production (capacity, costs, production speed) result from different technical conditions of the equipment used at the sites. Due to the option of manufacturing for other networks within the same planning horizon, the aforementioned costs can be influenced by existing opportunities. According to the principles of just-in-time production and delivery, stock-free material flows with no temporary storage between all sites of the network are assumed. Safety stocks, which are held for unexpected supply disruptions, remain untouched. Distribution is handled by third-party-logistic companies that provide appropriate means of transportation with fixed capacities (e.g., containers). As the sites and markets spread around the globe, individual shipments are preferable to delivery routes in general. Driving force of the networks' operations is the estimated demand at selected markets consisting of wholesalers and industrial customers. Besides finished products, even intermediates can be sold. As the final customers would be able to substitute their suppliers to a certain degree, incomplete demand satisfaction results in lost sales for the considered network. As mentioned before, we focus on networks of companies with limited access to capital markets, in particular small- and medium-sized enterprises with a small capital base. Due to the resulting lack of liquidity, pre-financing is required in order to initialize operations such as production or transportation. Since pre-financing is not taken on by the final customers due to the competitive situation, these companies are reliant on short-term credits, often characterized by high interest rates and daily interest calculation. In expectation of the revenues that are generated at the end of the multi-day planning horizon at the latest, even a series of follow-up financings can be required, whereas unbalanced payments during the same day can be compensated by bank overdrafts. In any case, it is necessary that the full financing amount is available at the start of each operation in order to run seamlessly. Due to the resulting financial obligations, the companies of the network are jointly forced to a thorough liquidity management on a daily basis in order to prevent insolvency.

The considered SC network consists of W subsequent production stages and one market stage (see Fig. 1). Each of these stages contains a certain number of sites. All sites within a particular production stage can be used to produce a certain quantity of a stage-specific good, which can be distributed to sites of a subsequent production stage, in order to be processed further (optionally in combination with goods from other SC stages) or to sites of the final market stage, in order to meet the given demand.

Fig. 1 Supply chain network structure



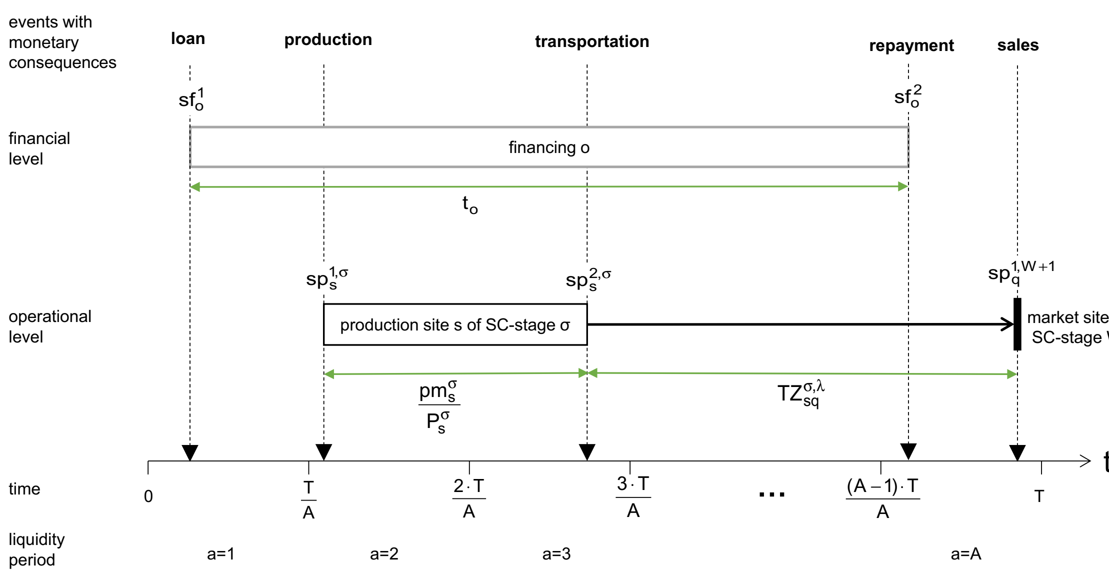


Fig. 2 Events, time points and liquidity periods

There are financing alternatives (potential financing arrangements such as short-term bank loans are available to the SC within the planning horizon and differ in repayment terms and credit rates), which serve to bridge SC operations and sales. These arrangements are characterized by the credit amount on the one hand, which is available to the SC at the point of time at which the contract enters into force, and the repayment amount including interest on the other hand, which is due and payable immediately after the specific term of the financing.

We assume a short-term planning horizon with a continuous time axis, which enables us to determine exact starting and ending times (i.e., specified points in time) of all relevant events concerning short-term SC planning (i.e., production, transportation, sales, loan, repayment). In addition, this planning horizon is subdivided into A liquidity periods (see Fig. 2). The latter are used to balance different monetary consequences, which are assigned to these periods completely at the starting times of production and transportation processes (production and transportation costs), at the time points of sale (revenue) and at the starting and ending times of realized financing alternatives (credit amounts and repayment amounts). In summary, the continuous-time scheduling of operations and financial transactions enables the planning of production, transportation, sales and financing exactly to the minute on the one hand, but additionally allows for day-by-day liquidity management within a given short-term multi-day planning horizon on the other hand.

The integrated MINLP model formulation combines the following decisions, while maximizing the profit of the entire SC *PR* at the end of the planning horizon and ensuring liquidity balancing within each liquidity period.

1. System-wide decisions at the operational level:
   - Operation of the sites being part of the SC within the planning horizon
   - Production and transportation quantities at/between the operating sites
   - Points in time at which production at each of the operating sites starts and ends
   - Points in time at which transportation between two operating sites starts and ends
   - Point in time at which the given demand of each of the markets is satisfied

2. System-wide decisions at the financial level:
   - Realization of available financing alternatives
   - Credit amount of each realized financing alternative
   - Points in time at which each realized financing alternative starts and ends

Simultaneous optimization ensures the coordination of the operational and the financial level. Thus, meeting the demand at the markets and initializing appropriate SC operations requires the realization of appropriate financings, and vice versa. Furthermore, there is a coordination between the points in time at which related events start and end.

Let $\sigma, \lambda \in \Gamma$ denote the SC stages of the network, which can be split up into W production stages ($\sigma = 1, \ldots, W$) and one market stage ($\sigma = W+1$). Within each of the SC stages, there are several sites $s, q \in S_\sigma$. The sites at the market stage are characterized by a given demand of products N_s^σ that can be satisfied by the network within the assumed planning horizon of T days in order to realize market-specific revenues EE_s^σ per unit sold. The selection and supplying of markets entails fixed marketing costs MK_s^{W+1}. As the products can be manufactured at production sites of all preceding SC stages $\sigma = 1, \ldots, W$, the sales of products with different levels of maturity (e.g., intermediate products, finished products) is considered. For each product manufactured in SC stage $\sigma = 2, \ldots, W$, exactly $B^{\lambda,\sigma}$ units of products manufactured in the preceding SC stages $\lambda = 1, \ldots, \sigma - 1$ are needed. After initialization of operations at a product site (that entails setup costs PF_s^σ), PC_s^σ units of products can be manufactured at the most due to given capacities. Manufacturing of products entails costs of PV_s^σ. There is a site-specific production speed P_s^σ that is measured in product units per day. The sites of the network are connected by potential material flows (Fig. 1). As one can see, each production site $s \in S_\sigma$ of a SC stage $\sigma = 1, \ldots, W$ is able to supply each site $q \in S_\lambda$ of a subsequent SC stage $\lambda = \sigma + 1, \ldots, W + 1$ that can be either a production site or a market. The maximum number of partial deliveries starting from a site $s \in S_\sigma$ and thus the maximum number of supplied sites of a subsequent SC stage, can be limited to V_s. If material flows are realized, related transports occur. They entail variable and fixed production costs of $TV_{sq}^{\sigma,\lambda}$ and $TF_{sq}^{\sigma,\lambda}$, respectively. According to the selected means of transportation, the maximum transportation capacity (measured in product units) is $TC_{sq}^{\sigma,\lambda}$ and the transportation time (measured in days) is $TZ_{sq}^{\sigma,\lambda}$. For bridging SC operations (production at the sites, transportation between the sites) and sales, financing alternatives $o \in O$ with different terms t_o (difference between the points in time, at which the financing alternative ends and starts) and credit rates i_o (measured in percent per day) are assumed to be available to the network. However, each of the alternatives is limited to the credit amount of FC_o. As it becomes necessary to balance the liquidity of the network within each of the A sub-periods $a \in \Lambda$ with the length of one day each, the planning horizon must be divided accordingly.

The decisions that need to be taken by the network managers can be detailed as follows. The SC partners strive for maximization of the profit *PR* at the end of the planning horizon. The latter requires optimal scheduling of starting points ($z = 1$) and end points ($z = 2$) of each of the aforementioned events related to SC operations and finance (see Fig. 2) within the given planning horizon. Firstly, both the time points $sp_s^{z,\sigma}$ must be determined for each production site that is operating for the network ($y_s^\sigma = 1$) and thus feeds in a production quantity of pm_s^σ ($pm_s^\sigma > 0$). If market sites are considered, the time point of meeting demand is only to be determined in case the market is supplied by the network. Between the network sites, transportation is required in order to ensure stock-free material flows to the markets. Conducted transports ($t_{sq}^{\sigma,\lambda} = 1$) result in fixed costs, and thus, they can be specified by the amount of products $x_{sq}^{\sigma,\lambda}$ ($x_{sq}^{\sigma,\lambda} > 0$) to be delivered. Consistently, non-conduction ($t_{sq}^{\sigma,\lambda} = 0$) is accompanied by $x_{sq}^{\sigma,\lambda} = 0$. Secondly, the two time points sf_o^z must be determined for each financing alternative that is used by the network in the amount of fi_o^z ($fi_o^z > 0$). In order to balance liquidity within each of the time periods, all the aforementioned time points $sp_s^{z,\sigma}$ and sf_o^z are assigned to their correspond-

Table 1 Fixation of auxiliary variables depending on event-period-assignment

	Assignment of event to liquidity period (e.g., $ep_{sa}^{1,\sigma}=1$)	Non-assignment of event to liquidity period (e.g., $ep_{sa}^{1,\sigma}=0$)
Constraints I [eg. (29)]	For the auxiliary (e.g., PK_{sa}^{σ}), the event-related liquidity consequences are set as the upper bound of the variable	
Constraints II [eg. (30)] in combination with non-negativity-constraints	For the auxiliary (e.g., PK_{sa}^{σ}), the event-related liquidity consequences are set as the lower bound of the variable	The auxiliary (e.g., PK_{sa}^{σ}) is greater than or equal to 0, i.e., the constraints are redundant in this case
Constraints III [eg. (31)] in combination with non-negativity-constraints	The auxiliary (e.g., PK_{sa}^{σ}) is less than or equal to C, i.e., the constraints are redundant in this case	The auxiliary (e.g., PK_{sa}^{σ}) is equal to 0
Results from combination of constraints I+II+III	The auxiliary (e.g., PK_{sa}^{σ}) is fixed to the event-related liquidity consequences (constraints I and II, upper bound = lower bound)	The auxiliary (e.g., PK_{sa}^{σ}) is fixed to 0 (constraints III)

ing liquidity period ($ep_{sa}^{z,\sigma}=1$ or $ef_{oa}^{z}=1$, respectively). Of course, each time point can only be assigned once, so there is formally a non-assignment ($ep_{sa}^{z,\sigma}=0$ or $ef_{oa}^{z}=0$, respectively) to all other liquidity periods. In case a production site is not chosen for operation ($y_s^{\sigma}=0$) and thus there is no production at it ($pm_s^{\sigma}=0$), the related time points $sp_s^{z,\sigma}$ are not interpretable, although they are part of the optimal solution. The same applies to market sites that are not supplied by the network, and analogously to the time points sf_o^{z}, if the related financing alternative is not used ($fi_o^{z}=0$).

The auxiliary variable L_a represents the liquidity surplus of a specific liquidity period. Due to liquidity balancing, it must be equal to zero in each liquidity period except the last one [see Eq. (3)]. Furthermore, the auxiliary variables ER_{sa}^{W+1}, PK_{sa}^{σ}, TK_{sa}^{σ} and FI_{oa}^{z} are required for linearization. They are equal to event-related liquidity consequences [i.e., to the variable monetary terms that are multiplied by binary assignment variables within the nonlinear constraints (2)], if an event is assigned to a liquidity period. Otherwise they are zero (see Table 1).

Model formulation

(i.) Objective Function and Liquidity Compensation

The objective function (1) maximizes the profit to be shared between the network companies and the network management at the end of the planning horizon.

$$\text{Max } PR \tag{1}$$

Profits are taken from liquidity surpluses generated by the network partners. Recall that liquidity is determined for all the liquidity periods (e.g., days) the planning horizon is divided into. According to Eq. (2), it is composed by monetary consequences of all the events that are assigned to a liquidity period. By the first term, marketing costs (that occur in case a market is selected) and revenues (that depend on the amount of products delivered to the market) are assigned to the period the market is supplied. By the second term, variable and fixed production costs are assigned to the period at which the manufacturing of goods starts at a production site. As system-wide stock-free material flows are assumed and hence production is directly followed by transportation, variable and fixed transportation costs are assigned to the period at which the production is finished in the third term. The fourth and fifth term assign credit amounts and repayment amounts to the periods at which the used financing alternatives start and end, respectively. Nonlinearity arises from the multiplication of binary assignment variables with monetary terms depending on decisions about SC operations and finance.

$$\begin{aligned}
&\sum_{s\in S_{W+1}} ep_{sa}^{1,W+1}\cdot\Bigg(-MK_s^{W+1}\cdot y_s^{W+1}\\
&\quad+\sum_{\lambda=1}^{W}\sum_{q\in S_\lambda} EE_s^{\lambda}\cdot x_{qs}^{\lambda,W+1}\Bigg)\\
&\quad-\sum_{\sigma=1}^{W}\sum_{s\in S_\sigma} ep_{sa}^{1,\sigma}\cdot\left(PV_s^{\sigma}\cdot pm_s^{\sigma}+PF_s^{\sigma}\cdot y_s^{\sigma}\right)\\
&\quad-\sum_{\sigma=1}^{W}\sum_{s\in S_\sigma} ep_{sa}^{2,\sigma}\cdot\Bigg(\sum_{\lambda=\sigma+1}^{W+1}\sum_{q\in S_\lambda} TV_{sq}^{\sigma,\lambda}\cdot x_{sq}^{\sigma,\lambda}\\
&\quad+TF_{sq}^{\sigma,\lambda}\cdot t_{sq}^{\sigma,\lambda}\Bigg)+\sum_{o\in O} ef_{oa}^{1}\cdot fi_o^{1}\\
&\quad-\sum_{o\in O} ef_{oa}^{2}\cdot fi_o^{2}\geq L_a;\quad \forall a=1,\ldots,A
\end{aligned}\tag{2}$$

In order to prevent insolvency of the network, it is necessary to balance all the monetary consequences that are assigned to a specific liquidity period by $L_a=0$ in Eq. (3). The liquidity surplus of the last liquidity period equals the profit to be maximized in the objective function.

$$L_a = \begin{cases} 0 & \forall a = 1, \ldots, A-1 \\ PR & \forall a = A \end{cases} \quad (3)$$

(ii.) Operations and Finance Module

The driving force of SC operations is the demand for different goods (intermediate products and finished products) at the markets, which need not be satisfied in full [Eq. (4)]. The goods are obtained from different production stages due to their different levels of maturity. A prerequisite is that the market is selected for delivery.

$$\sum_{q \in S_\lambda} x_{qs}^{\lambda,W+1} \leq N_s^\lambda \cdot y_s^{W+1}; \quad \forall \lambda = 1, \ldots, W; s \in S_{W+1} \quad (4)$$

In order to manufacture goods at the production sites, specific quantities of goods from sites of the previous stages are required according to given bills of materials [Eq. (5)].

$$\sum_{q \in S_\lambda} x_{qs}^{\lambda,\sigma} = B^{\lambda,\sigma} \cdot pm_s^\sigma; \quad \forall \lambda = 1, \ldots, \sigma-1; s \in S_\sigma; \sigma = 2, \ldots, W \quad (5)$$

The continuity of material flows is guaranteed by Eq. (6) that limits the sum of outgoing transportation quantities (i.e., quantities that are delivered to production sites or markets of subsequent SC stages) to the production quantity at a production site.

$$\sum_{\lambda=\sigma+1}^{W+1} \sum_{q \in S_\lambda} x_{sq}^{\sigma,\lambda} \leq pm_s^\sigma; \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W \quad (6)$$

In case that it is produced at a site or it is transported between different sites, respectively, the given maximum capacities need to be respected according to Eqs. (7) and (8).

$$pm_s^\sigma \leq PC_s^\sigma \cdot y_s^\sigma; \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W \quad (7)$$

$$x_{sq}^{\sigma,\lambda} \leq TC_{sq}^{\sigma,\lambda} \cdot t_{sq}^{\sigma,\lambda}; \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W; q \in S_\lambda; \lambda = \sigma+1, \ldots, W+1 \quad (8)$$

If an available financing alternative is used to bridge SC operations and sales, the amount of credit, including interest, must be repaid in full at the end of the specified term. Interest calculation on a daily basis is assumed [Eq. (9)].

$$fi_o^1 \cdot (1+i_o)^{t_o} = fi_o^2; \quad \forall o \in O \quad (9)$$

(iii.) Scheduling Module

Scheduling requires the exact determination of start times and end times of production. According to Eq. (10), the difference between both points in time is the production time that can be calculated by the quotient of production quantity and production speed.

$$sp_s^{1,\sigma} + \frac{pm_s^\sigma}{P_s^\sigma} = sp_s^{2,\sigma}; \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W \quad (10)$$

If two production sites of subsequent SC stages are connected by some form of transports ($t_{sq}^{\sigma,\lambda} = 1$), the difference between the production start time of one site and the production end time of the other site, belonging to a preceding SC stage, is fixed to the relevant transportation time, as $sp_s^{2,\sigma} + TZ_{sq}^{\sigma,\lambda} = sp_q^{1,\lambda}$ is valid in this case. If transports between production sites and markets are considered alternatively, times of meeting given demand are to be used analogously instead of production start times. If there is no connection between two sites ($t_{sq}^{\sigma,\lambda} = 0$), the time points resulting at the right-hand side of both Eqs. (11) and (12) will be out of the range of the planning horizon due to the addition/subtraction of the big number C. Then, both the constraints become redundant.

$$sp_s^{2,\sigma} + TZ_{sq}^{\sigma,\lambda} \leq sp_q^{1,\lambda} + C \cdot (1 - t_{sq}^{\sigma,\lambda}); \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W; q \in S_\lambda; \lambda = \sigma+1, \ldots, W+1 \quad (11)$$

$$sp_s^{2,\sigma} + TZ_{sq}^{\sigma,\lambda} \geq sp_q^{1,\lambda} - C \cdot (1 - t_{sq}^{\sigma,\lambda}); \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W; q \in S_\lambda; \lambda = \sigma+1, \ldots, W+1 \quad (12)$$

Equation (13) ensures that the time of meeting demand at the markets cannot exceed the end of the planning horizon.

$$sp_s^{1,W+1} \leq T; \quad \forall s \in S_{W+1} \quad (13)$$

The start and end times of financial transactions are linked in Eq. (14) by the specified financing terms that are given for each financing alternative.

$$sf_o^1 + t_o = sf_o^2; \quad \forall o \in O \quad (14)$$

All events leading to monetary consequences must be assigned to exactly one of the A liquidity periods into which the planning horizon with the length of T days is divided. By the Eqs. (15) and (16), respectively, the intervals of the liquidity periods $a = 1, \ldots, A$ (see Fig. 2) are defined as $\left[0; \frac{T}{A}\right), \left[\frac{T}{A}; \frac{2 \cdot T}{A}\right), \left[\frac{2 \cdot T}{A}; \frac{3 \cdot T}{A}\right), \ldots, \left[\frac{(A-1) \cdot T}{A}; T\right]$. The lower and upper bounds of these intervals are restrictive for each of the time points $sp_s^{z,\sigma}$ associated with SC operations (Eq. 15), if the related event is assigned to the liquidity period ($ep_{sa}^{z,\sigma} = 1$). In case of non-assignment ($ep_{sa}^{z,\sigma} = 0$), the constraints become redundant due to the addition/subtraction of the big number C. Analogously, the time points sf_o^z associated with the usage of financing alternatives (Eq. 16) are matched with the intervals of the liquidity periods. In order to ensure that the time point T is part of the planning horizon $[0, T]$ and thus also part of its last liquidity period, both the less-than signs must be replaced by less-than-or-equal signs in the constraints (15) and (16) for $a = A$.

$$(a-1)\cdot\frac{T}{A} - C\cdot(1-ep_{sa}^{z,\sigma}) \le sp_s^{z,\sigma} < a\cdot\frac{T}{A} + C\cdot(1-ep_{sa}^{z,\sigma}); \quad \forall z\in Z; s\in S_\sigma; \sigma=1,\ldots,W+1; a=1,\ldots,A \tag{15}$$

$$(a-1)\cdot\frac{T}{A} - C\cdot(1-ef_{oa}^{z}) \le sf_o^{z} < a\cdot\frac{T}{A} + C\cdot(1-ef_{oa}^{z}); \quad \forall z\in Z; o\in O; a=1,\ldots,A \tag{16}$$

The assignment of an event to a certain liquidity period must be enforced in case that event occurs. If a production site is in operation or a market site is supplied, the production start time or the time of meeting demand, respectively, must be assigned to (at least) one of the liquidity periods due to Eq. (17). The transportation start time at a site (that is equal to the production end time at this site) must be assigned to (at least) one liquidity period, if at least one transport starts from the production site. In this context, Eq. (18) additionally ensures that a maximum number of partial deliveries starting from the production site is not exceeded. Equation (19) assigns the start time of a used financing alternative to (at least) one liquidity period and ensures that the related credit amount does not exceed a given credit limit. Recall that it is not possible for a production start time, transportation start time or financing start time to be assigned to more than one liquidity period due to Eqs. (15) and (16), and thus, a unique assignment is ensured.

$$y_s^\sigma \le \sum_{a\in\Lambda} ep_{sa}^{1,\sigma}; \quad \forall s\in S_\sigma; \sigma=1,\ldots,W+1 \tag{17}$$

$$\sum_{\lambda=\sigma+1}^{W+1}\sum_{q\in S_\lambda} t_{sq}^{\sigma,\lambda} \le V_s\cdot\sum_{a\in\Lambda} ep_{sa}^{2,\sigma}; \quad \forall s\in S_\sigma; \sigma=1,\ldots,W+1 \tag{18}$$

$$fi_o^1 \le FC_o\cdot\sum_{a\in\Lambda} ef_{oa}^{z}; \quad \forall z\in Z; o\in O \tag{19}$$

In order to avoid the unnecessary assignment of nonevents by Eqs. (17)–(19), additional constraints can be added for strengthening the model formulation.

Finally, it should be remarked that the variables related to the event-period-assignment and the variables indicating the usage of sites and the conduction of transports must be stated as belonging to the domain of binaries. All other variables are assumed to be continuous.

$$ef_{oa}^{z} \in \{0;1\}; \quad \forall z\in Z; o\in O; a\in\Lambda \tag{20}$$

$$ep_{sa}^{z,\sigma} \in \{0;1\}; \quad \forall z\in Z; \sigma=1,\ldots,W+1; s\in S_\sigma; a\in\Lambda \tag{21}$$

$$t_{sq}^{\sigma,\lambda} \in \{0;1\}; \quad \forall \sigma=1,\ldots,W; \lambda=\sigma+1,\ldots,W+1; s\in S_\sigma; q\in S_\lambda \tag{22}$$

$$y_s^\sigma \in \{0;1\}; \quad \forall\sigma=1,\ldots,W+1; s\in S_\sigma \tag{23}$$

$$fi_o^z, sf_o^z \ge 0; \quad \forall z\in Z; o\in O \tag{24}$$

$$pm_s^\sigma \ge 0; \quad \forall\sigma=1,\ldots,W; s\in S_\sigma \tag{25}$$

$$sp_s^{z,\sigma} \ge 0; \quad \forall z\in Z; \sigma=1,\ldots,W+1; s\in S_\sigma \tag{26}$$

$$x_{sq}^{\sigma,\lambda} \ge 0; \quad \forall\sigma=1,\ldots,W; \lambda=\sigma+1,\ldots,W+1; s\in S_\sigma; q\in S_\lambda \tag{27}$$

$$PR \ge 0; \tag{28}$$

4 Linearization

In order to reduce computational efforts for the optimization, the mixed-integer nonlinear program (1)–(28) is transformed into a mixed-integer linear program. Due to the structure of Eq. (2), the monetary consequence of an operational or financial event only affects liquidity in the period it is assigned to by using binary variables. Recall that the event-period-assignment is part of the optimization. As the monetary consequences of assigned events need to be quantified by other continuous and/or binary variables as further part of the optimization, nonlinearity arises in each term of Eq. (2). The following linearization contains a reformulation of the latter constraints and results in an *equivalent* model formulation with identical objective values (see Sect. 5). It is based on an alternative use of the binary variables indicating the event-period-assignment in the context of the Big M method, which requires the introduction of the continuous nonnegative auxiliary variables ER_{sa}^{W+1}, PK_{sa}^{σ}, TK_{sa}^{σ} and FI_{oa}^{z}.

The latter variables represent (site- or object-specific) monetary consequences of a certain event type (e.g., production costs at the time point of production start), if the time point of the event's occurrence belongs to the liquidity period that is connected to the auxiliary variable by the period-specific index. Thus, it must be ensured that each of the auxiliary variables equals the (optimized) monetary consequences of its related event, if the event is (optimally) assigned to the liquidity period. Otherwise, the auxiliary variable must be equal to zero. For this purpose, the auxiliary variables need to be linked to the terms of the former liquidity constraints (2) that were quantifying monetary consequences of occurring events. This is realized by three additional constraints I, II, III for each of the four event types (sales, production, transportation, and financing), which are part of the new linearized model formulation.

Due to the combination of constraints I, II, III for each type of event, the auxiliary variables are fixed to specific values that depend on the event-period-assignment (see Table 1). Because of structural analogies, only the constraints of the production event are exemplified in the following.

Equation (29) limits the auxiliaries PK_{sa}^{σ} to their upper bound, i.e., to the monetary consequences that would affect the liquidity of a period, if the event is assigned to it. In particular, the sum of fixed and variable production costs at a specific network site builds the upper bound of the site's auxiliary variables PK_{sa}^{σ} for each liquidity period.

$$PK_{sa}^{\sigma} \leq PV_s^{\sigma} \cdot pm_s^{\sigma} + PF_s^{\sigma} \cdot y_s^{\sigma}; \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W; a \in \Lambda \tag{29}$$

Equation (30) defines the lower bound of the auxiliaries PK_{sa}^{σ}. In case the production event is assigned to a period ($ep_{sa}^{1,\sigma} = 1$), the term $-C \cdot (1 - ep_{sa}^{1,\sigma})$ is equal to zero. Hence, the production costs [being equal to the auxiliary's upper bound, see Eq. (29)] are set as lower bound of the auxiliary variable that represents this specific liquidity period. In the alternative case ($ep_{sa}^{1,\sigma} = 0$), Eq. (30) becomes redundant for the site's auxiliary variables (that represent other liquidity periods), as these constraints set a lower bound that is dominated by the one resulting from the auxiliary variables' nonnegativity constraints (32).

$$PK_{sa}^{\sigma} \geq PV_s^{\sigma} \cdot pm_s^{\sigma} + PF_s^{\sigma} \cdot y_s^{\sigma} - C \cdot (1 - ep_{sa}^{1,\sigma}); \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W; a \in \Lambda \tag{30}$$

Equation (31) is only restrictive for auxiliaries PK_{sa}^{σ} that represent liquidity periods without any assigned events ($ep_{sa}^{1,\sigma} = 0$). Then, the upper bound of zero [being equal to the auxiliaries' lower bound, see Eq. (32)] is valid. In the other case ($ep_{sa}^{1,\sigma} = 1$), Eq. (31) becomes redundant, as the upper bound of PK_{sa}^{σ}, which is set by Eq. (29), dominates the one that results from $C \cdot ep_{sa}^{1,\sigma}$.

$$PK_{sa}^{\sigma} \leq C \cdot ep_{sa}^{1,\sigma}; \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W; a \in \Lambda \tag{31}$$

$$PK_{sa}^{\sigma} \geq 0; \quad \forall s \in S_\sigma; \sigma = 1, \ldots, W; a \in \Lambda \tag{32}$$

Due to the combination of the auxiliaries' lower and upper bounds that are set by Eqs. (29)–(32), it becomes obvious that the assignment of a site's event to a liquidity period ($ep_{sa}^{1,\sigma} = 1$) results in $PK_{sa}^{\sigma} = PV_s^{\sigma} \cdot pm_s^{\sigma} + PF_s^{\sigma} \cdot y_s^{\sigma}$, and a non-assignment ($ep_{sa}^{1,\sigma} = 0$) in $PK_{sa}^{\sigma} = 0$. In other words, one can see that the auxiliary variable is fixed to the monetary consequences in case of the event-period-assignment; otherwise, it is fixed to zero. Considering the remaining events analogously while calculating a period's liquidity, Eq. (2) can be replaced by Eq. (33).

$$\sum_{s \in S_{W+1}} ER_{sa}^{W+1} - \sum_{\sigma=1}^{W} \sum_{s \in S_\sigma} PK_{sa}^{\sigma} - \sum_{\sigma=1}^{W} \sum_{s \in S_\sigma} TK_{sa}^{\sigma} + \sum_{o \in O} FI_{oa}^{1} - \sum_{o \in O} FI_{oa}^{2} \geq L_a; \quad \forall a = 1, \ldots, A \tag{33}$$

As liquidity is affected by the events exactly as modeled in the nonlinear liquidity constraints (2), the equivalence of the linearized formulation can be claimed in each case that the parameter C is chosen appropriately, i.e., that the parameter exceeds the maximum monetary consequences that could be caused by any event.

5 Numerical analysis

The modeling was used to optimize a three-stage SC ($W = 2$) with three sites in each stage. A planning horizon of 10 days, which was split into 10 liquidity periods for day-by-day liquidity compensation, was assumed. Furthermore, 10 financing alternatives with the following conditions (see Table 2) are available for the SC.

All other data (see Tables 3, 4, 5) were generated randomly by the optimization software package GAMS 24.5.4, using an uniform distribution with specified intervals. Between both the production stages, a production coefficient of 0.2 is assumed.

For the computations, a cluster of the University of Greifswald was used. It consists of 78 machines, with two Intel Xenon E5-2623 v3 Quad-Core-CPUs, 3.00 GHz, 8 GT/s and 64 GB RAM in each of them. Consequently, 16 threads per machine are available. For each of the calculations, up to eight machines were harnessed simultaneously.

The model was implemented in GAMS 24.5.4. As numerical methods of finite-precision arithmetic are usually used by commercial solvers, tolerance parameters defining the feasibility of a solution must be set before starting the optimization. This is especially important for models with numerical difficulties, which arise from using Big M coefficients (GAMS 2015). This coefficient is part of the schedul-

Table 2 Financing parameters

	Term of the financing (days)	Credit rate (% per day)
Financings 1 and 2	2	0.1
Financings 3 and 4	3	0.2
Financings 5 and 6	4	0.3
Financings 7 and 8	5	0.4
Financings 9 and 10	6	0.5

Table 3 Transport parameters

Variable transportation costs ($/unit) fixed transportation costs ($) transportation capacity (units) transportation time (days)	To site 2,1	To site 2,2	To site 2,3	To site 3,1	To site 3,2	To site 3,3
From site 1,1	12	11	12	11	12	8
	132	127	54	117	122	60
	106	195	125	88	109	200
	1.3	0.9	0.6	1.3	0.8	0.9
From site 1,2	12	8	10	12	10	11
	62	122	59	70	105	149
	186	69	180	62	161	179
	1.5	0.8	0.9	1.2	1.2	1.2
From site 1,3	9	9	12	8	12	10
	100	116	74	71	51	117
	135	163	136	68	169	72
	0.7	0.6	1	0.7	0.9	0.6
From site 2,1				12	10	8
				99	88	91
				55	86	92
				1.1	1.1	0.7
From site 2,2				9	9	11
				98	86	59
				120	114	133
				1.2	1.1	1.2
From site 2,3				10	8	9
				123	123	94
				104	111	162
				1.2	0.7	0.6

Table 4 Production parameters

	Site 1,1	Site 1,2	Site 1,3	Site 2,1	Site 2,2	Site 2,3
Variable production costs ($/unit)	80	103	88	82	119	108
Fixed production costs ($)	1057	1025	1415	885	1230	745
Production capacity (units)	155	164	178	197	120	124
Production speed (units/day)	7	8	7	8	10	10

Table 5 Sales parameters

	Site 3,1	Site 3,2	Site 3,3
Demand for goods produced in SC stage $\sigma = 1$ (units)	94	77	58
Demand for goods produced in SC stage $\sigma = 2$ (units)	56	52	86
Revenue for goods produced in SC stage $\sigma = 1$ ($/unit)	402	439	448
Revenue for goods produced in SC stage $\sigma = 2$ ($/unit)	431	426	437
Fixed marketing costs ($)	150	124	54

ing constraints (11) and (12), the constraints (15) and (16) used for the assignment of event times to liquidity periods, as well as the linearization constraints II and III [i.e., Eqs. (30) and (31) for production event]. In each case, $C = 1{,}000{,}000$ was chosen. Due to the combination of the coefficient with binary values within the constraints, the feasibility tolerance was reduced to 10^{-8}.

First, the SCIP 3.2 solver, which is able to deal with nonlinear models on a single machine of the cluster by applying an interior point optimizer (GAMS 2015), was used to compute the nonlinear model (1)–(28). However, an optimal solution could not be obtained until the computation terminated after the time limit of 24 h. Although a feasible solution of $ 74,366 was found, a high relative gap (i.e., percentage difference between the primal bound and the dual bound) of 101.71 % was valid at the same time.

Using the linearized model formulation (1), (3)–(33) the benefits of using commercial solvers applicable to MILP

Table 6 Liquidity balancing

Liquidity period	$a = 1$	$a = 4$	$a = 5$	$a = 7$	$a = 9$	$a = 10$
Production						
Site 1,1	−6209.0					
Site 1,2	−8276.2					
Site 1,3	−3720.6					
Site 2,1			−4071.3			
Site 2,2			−6653.0			
Site 2,3			−5774.7			
Transportation						
Sites 1,1–3,2						−894.8
Sites 1,2–3,2					−229.0	
Sites 1,2–3,3					−787.0	
Sites 1,3–2,1		−169.9				
Sites 1,3–2,2		−198.0				
Sites 1,3–2,3		−185.8				
Sites 2,1–3,3						−401.9
Sites 2,2–3,2					−496.1	
Sites 2,3–3,3						−513.1
Sales						
Site 3,1						0.0
Site 3,2						53,128.4
Site 3,3						63,316.0
Marketing costs						−178.0
Financings						
Financing 1					1512.1	−1515.2
Financing 2			34,924.3	−34,994.1		
Financing 3		553.7		−557.1		
Financing 4				35,551.2		−35,764.9
Financing 5	18,205.8		−18,425.3			
Balance	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**77,176.6**

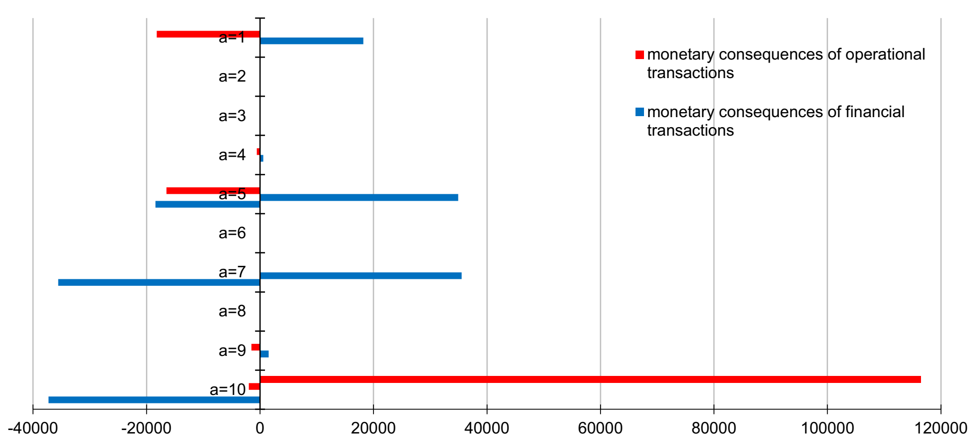


Fig. 3 Monetary consequences within liquidity periods

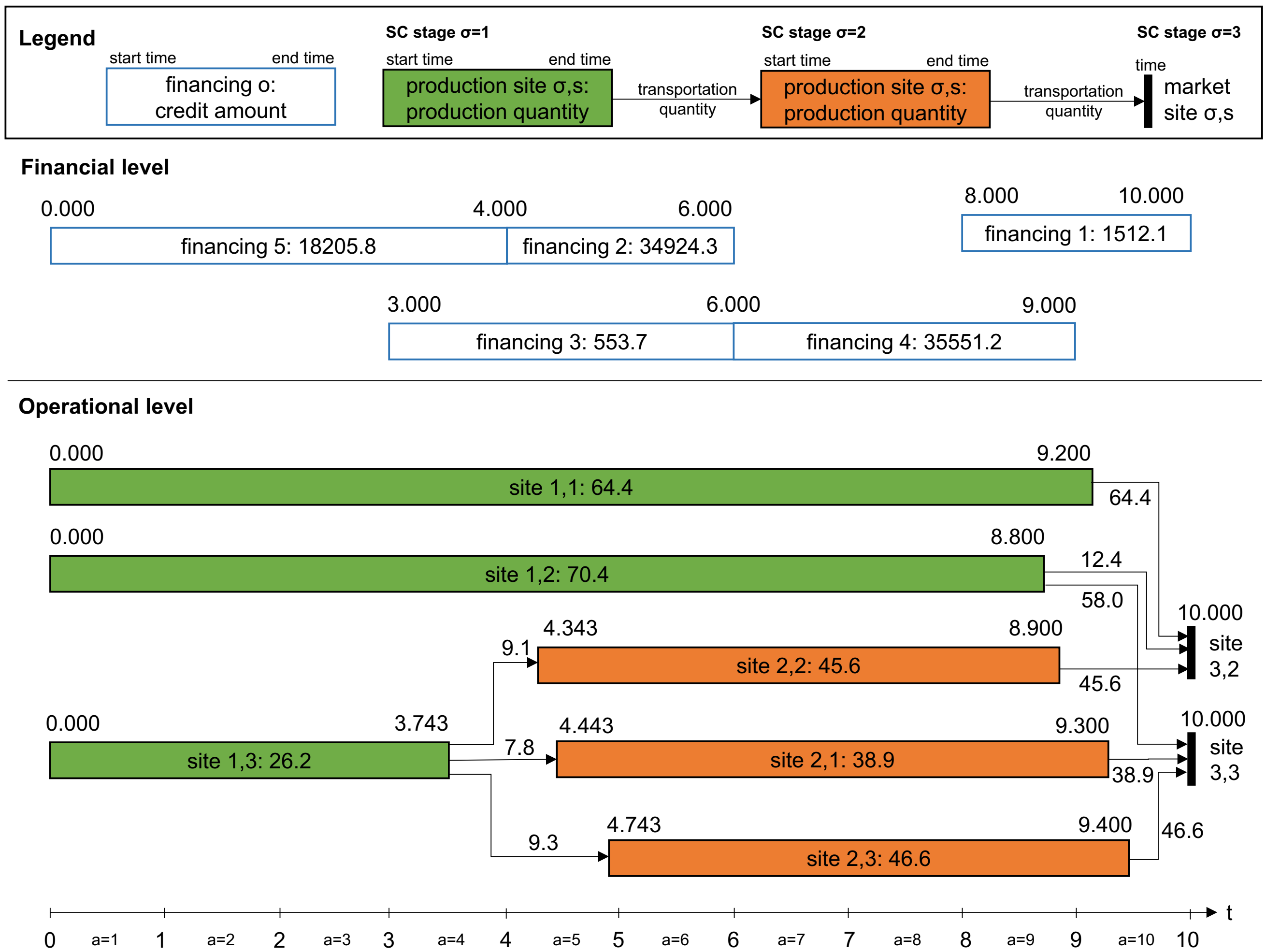


Fig. 4 Optimal planning and scheduling

problems can be realized. First, the CPLEX 12.6.2 solver was applied. Differing from the default settings, the software uses the maximum number of threads. As the search tree to be processed by shared-memory branch and bound resides on a single machine of the cluster, the related computation is comparable to a run on a conventional workstation. The proven optimal solution of $ 77,177 was found after 3 h and 9 min.

In order to make use of the benefits of the cluster, the computations were repeated with the CPLEX-D 12.6.2 solver. Opposite to the regular version of the solver, it enables computing difficult MILP problems in a distributed way by utilizing more than one machine of the cluster (GAMS 2015). In particular, the model runs on a single master associated with multiple workers. As the solver harnesses the power of multiple machines, it is possible to achieve better performance. There are different modes of optimization that can be selected before the start of calculation by specific settings. The first alternative is *Concurrent Mode*. During the computation each worker of the cluster applies different parameter settings to the same problem as the other workers. In contrast, *Distributed Mode* is possible. If it is applied to the model, each worker computes one specific part of a common search tree and communicates its findings to the master. The latter is responsible for the coordination of the workers. Even a combination of the aforementioned modes of the CPLEX-D solver is possible. Within the *Combined Mode*, the solver starts in the Concurrent Mode. The latter stops automatically, if a sufficiently large search tree was created by at least one of the workers. After that, the master selects the worker with the best performance so far and uses its search tree for the optimization in the subsequent Distributed Mode. All three modes have been tested for the linear model (1), (3)–(33) by using eight machines (with 16 threads each) of the cluster. As a result, we found out that only the CPLEX-D solver in the Concurrent Mode was able to solve the MILP to optimality within a time limit of 24 h. Here, the maximum profit of $ 77,177 was calculated and proven after 27 min and 6 s. It can be assumed that the CPLEX-D solver in the Distributed Mode (gap of 57.37 % after 24 h) and in the Combined Mode (gap

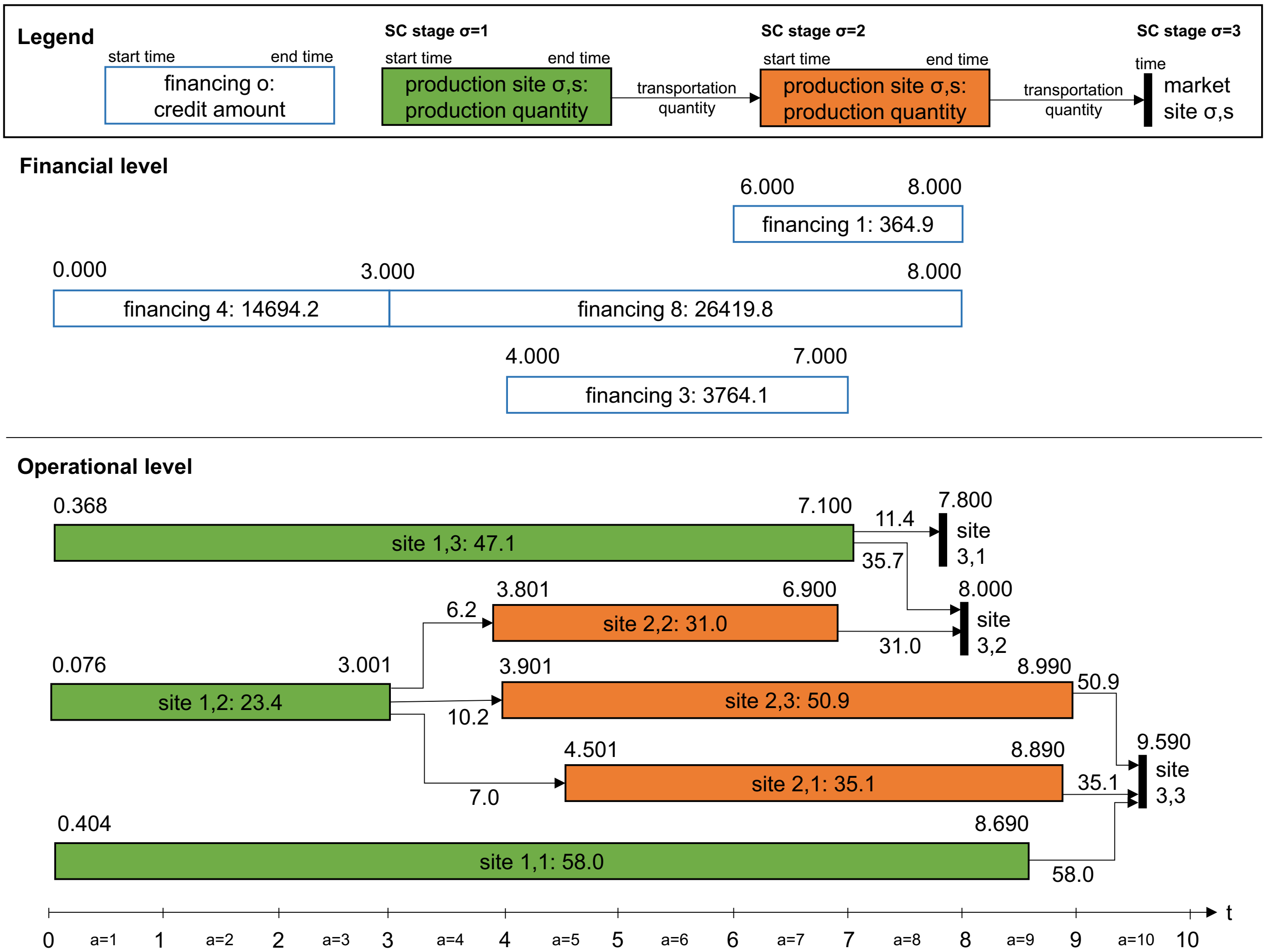


Fig. 5 Optimal planning and scheduling with time window [7.5; 8.5] for meeting demand at sites 3,1 and 3,2

of 54.12 % after 24 h) failed due to an inefficient coordination of the workers with regard to our specific problem structure.

Whenever the optimal solution was found, the optimal events of production, transportation, sales and financings are assigned to six of the ten liquidity periods, as shown in Table 6. The monetary consequences of all operational transactions (i.e., costs, revenue) and financial transactions (i.e., credit amounts and repayment amounts) are balanced within these periods (Fig. 3), until the maximum profit is obtained at the end of the planning horizon.

Within the optimal solution, liquidity management as well as SC planning and scheduling are matched to each other (Fig. 4). In our case, loans with short maturities and low interest rates are preferred. A four-day loan at the beginning of planning horizon is used to initiate operations. Due to these financial resources, all three available sites of the first SC stages can start manufacturing at the same point in time. Whereas both sites 1,1 and 1,2 produce exclusively for direct market orders, the products of site 1,3 are used as intermediates for the following SC stage. As transportation of these intermediates begins in the fourth liquidity period (i.e., immediately after the production of site 1,3 ends), additional money must be borrowed. The same applies to the following liquidity period, when the production of all three sites belonging to the second SC stage (sites 2,1–2,3) starts. Additionally, a refinancing of the initial four-day loan is required. The seventh liquidity period is solely characterized by refinancing transactions, i.e., two loans expiring at the same point in time are replaced by one new three-day loan. The ninth liquidity period covers the end of production at two different sites. Thus, another loan is required to finance subsequent transports of finished products to the market stage. With respect to the sites' production times (depending on their optimal production quantity) and the given transportation times, both markets 3,2 and 3,3 are served exactly at the end of the planning horizon to a certain percentage, whereas the market at site 3,1 cannot be supplied. Revenue of the last liquidity period is used to compensate three further transports, the marketing costs and all remaining repayment amounts.

Table 7 Results of the scenario analysis

No.	SCIP 3.2 solver (16 threads)			CPLEX 12.6.2 solver (16 threads)			CPLEX-D 12.6.2 solver concurrent mode (8 × 16 threads)			CPLEX-D 12.6.2 solver distributed mode (8 × 16 threads)			CPLEX-D 12.6.2 solver combined mode (8 × 16 threads)		
	Objective value ($)	Gap (%)	Comp.time (hh:mm:ss)	Objective value ($)	Gap (%)	Comp.time (hh:mm:ss)	Objective value ($)	Gap (%)	Comp.time (hh:mm:ss)	Objective value ($)	Gap (%)	Comp.time (hh:mm:ss)	Objective value ($)	Gap (%)	Comp.time (hh:mm:ss)
1	63, 512	0	3:44:52[a]	63, 512	0	0:00:40	63, 512	0	0:00:15	63, 512	0	0:01:43	63, 512	0	0:00:15
2	36, 299	60.1	2:00:00*	48, 984	0	0:00:52	48, 984	0	0:04:10	48, 868	15.9	2:00:00*	48, 984	13.8	2:00:00*
3	48, 811	41.4	2:00:00*	65, 421	0	0:02:02	65, 421	0	0:00:24	65, 421	5.2	2:00:00*	65, 421	0	0:10:37
4	48, 192	40.4	2:00:00*	60, 141	0	0:50:45	60, 141	0	0:00:51	60, 141	11.1	2:00:00*	59, 634	11.8	2:00:00*
5	67, 082	37.0	2:00:00*	67, 107	0	0:10:09	67, 107	0	0:00:40	67, 018	25.5	2:00:00*	67, 107	25.5	2:00:00*
6	64, 237	24.2	2:00:00*	64, 237	0	0:31:07	64, 237	0	0:02:53	58, 631	24.9	2:00:00*	64, 237	7.4	2:00:00*
7	38, 835	54.5	2:00:00*	44, 004	0	0:55:56	44, 004	0	0:00:48	42, 095	29.8	2:00:00*	44, 001	26.3	2:00:00*
8	55, 472	44.0	2:00:00*	60, 856	0	1:46:43	60, 856	0	0:08:38	60, 856	23.1	2:00:00*	60, 802	17.9	2:00:00*
9	65, 546	20.1	2:00:00*	75, 565	0	0:01:16	75, 565	0	0:00:52	75, 565	4.0	2:00:00*	75, 034	4.4	2:00:00*
10	70, 230	10.5	2:00:00*	74, 937	0	0:03:10	74, 937	0	0:00:41	72, 071	7.1	2:00:00*	74, 933	3.4	2:00:00*
11	71, 692	36.2	2:00:00*	72, 865	0	1:30:50	72, 865	0	0:06:40	71, 364	26.9	2:00:00*	70, 195	28.0	2:00:00*
12	48, 960	18.4	2:00:00*	54, 992	0	0:10:28	54, 992	0	0:02:48	53, 486	7.7	2:00:00*	53, 005	8.5	2:00:00*
13	44, 127	73.1	2:00:00*	51, 578	0	0:27:21	51, 578	0	0:12:20	49, 013	41.4	2:00:00*	48, 387	37.6	2:00:00*
14	43, 511	64.5	2:00:00*	54, 943	0	1:15:37	54, 943	0	0:06:13	54, 878	23.3	2:00:00*	53, 695	24.7	2:00:00*
15	39, 962	74.5	2:00:00*	43, 943	0	0:17:38	43, 943	0	0:12:43	43, 351	35.5	2:00:00*	43, 202	29.6	2:00:00*
16	57, 791	46.5	2:00:00*	60, 175	0	0:02:15	60, 175	0	0:00:49	56, 162	38.5	2:00:00*	57, 958	19.2	2:00:00*
17	48, 136	59.5	2:00:00*	50, 366	0	0:30:18	50, 366	0	0:02:58	49, 081	37.5	2:00:00*	50, 080	36.3	2:00:00*
18	29, 414	136.6	2:00:00*	42, 778	0	1:35:29	42, 778	0	0:09:15	41, 325	47.5	2:00:00*	42, 520	40.4	2:00:00*
19	36, 695	62.8	2:00:00*	42, 930	0	1:21:29	42, 930	0	0:01:28	42, 586	23.2	2:00:00*	42, 251	28.8	2:00:00*
20	86, 616	84.2	2:00:00*	86, 716	29.5	2:00:00*	86, 716	0	0:00:53	84, 708	41.5	2:00:00*	86, 319	39.5	2:00:00*
21	43, 288	66.4	2:00:00*	53, 619	21.0	2:00:00*	53, 632	0	0:05:27	51, 446	25.0	2:00:00*	50, 378	22.6	2:00:00*
22	83, 225	72.3	2:00:00*	83, 899	27.2	2:00:00*	83, 899	0	0:07:25	83, 753	38.3	2:00:00*	83, 665	37.3	2:00:00*
23	80, 031	60.8	2:00:00*	80, 031	0	0:37:19	80, 031	0	0:29:10	78, 515	29.2	2:00:00*	79, 695	30.8	2:00:00*
24	80, 545	141.0	2:00:00*	88, 631	34.3	2:00:00*	88, 631	0	0:15:53	84, 106	58.8	2:00:00*	88, 358	53.7	2:00:00*
25	109, 905	83.0	2:00:00*	114, 506	25.8	2:00:00*	114, 506	0	0:23:54	114, 506	49.2	2:00:00*	112, 562	45.2	2:00:00*
26	99, 140	91.2	2:00:00*	103, 034	24.7	2:00:00*	103, 034	0	0:34:59	101, 414	51.7	2:00:00*	101, 483	46.6	2:00:00*
27	81, 952	117.0	2:00:00*	100, 850	27.0	2:00:00*	100, 850	0	0:12:46	86, 651	54.6	2:00:00*	100, 484	43.2	2:00:00*
28	94, 427	105.5	2:00:00*	101, 306	28.0	2:00:00*	101, 306	0	0:06:39	94, 138	51.5	2:00:00*	94, 898	51.1	2:00:00*
29	48, 012	247.5	2:00:00*	70, 057	0	0:38:15	70, 057	0	1:15:51	69, 407	55.1	2:00:00*	69, 418	55.2	2:00:00*
30	82, 059	79.5	2:00:00*	82, 321	29.3	2:00:00*	82, 321	0	0:15:24	78, 195	48.0	2:00:00*	82, 245	48.9	2:00:00*

*Computation was terminated after 2 h

[a] Computation would result in a feasible objective value of 63,504 (gap 0.0126 %) if it would be terminated after 2 h

Table 8 Statistical results of the scenario analysis

	SCIP 3.2 (16 threads)	CPLEX 12.6.2 (16 threads)	CPLEX-D 12.6.2 concurrent mode (8 × 16 threads)	CPLEX-D 12.6.2 distributed mode (8 × 16 threads)	CPLEX-D 12.6.2 combined mode (8 × 16 threads)
Number of optimal solutions found within 2 h	0	21	30	1	2
Maximum gap	247.5 %	34.3 %	0 %	58.8 %	55.2 %
Average gap	68.4 %	8.2 %	0 %	31.0 %	27.9 %

In addition to the aforementioned optimal solution, time windows of [7.5; 8.5] were prescribed to meet demand at the markets 3,1 and 3,2. This led to a complete restructuring of the previous optimal solution, as shown in Fig. 5. Consequently, a lower maximal profit of $ 63,512 (see Table 7, scenario No. 1) can be obtained. Although the optimal number of financings is reduced, all three markets are (partly) supplied with goods produced in both SC stages. All sales are terminated before the end of the planning horizon. Furthermore, it is noteworthy that financing is only required until the eighth day of the planning horizon, because revenue from the two markets can be used to finance transportation activities within the remaining days of the planning horizon.

Finally, it should be shown that the proposed equivalent linearization (1), (3)–(33) is advantageous going beyond the aforementioned numerical example. For this reason, a scenario analysis was conducted. The same SC structure (three-stage SC with three sites in each stage, planning horizon of 10 days, 10 liquidity periods, 10 financing alternatives with given terms) was used to generate 30 different scenarios, which are based on different data sets. The latter were randomly generated within given intervals by changing the seed parameter in the source code of our model (GAMS 2015). In some cases, time windows for meeting the given demand were set. The calculations were performed with the same aforementioned high-performance hardware. According to the previous illustrative example, each of the scenarios was computed with the SCIP 3.2 and the CPLEX 12.6.2 solver, which are limited to the use of one machine of the cluster. Thus, the results are comparable to those that could be generated on a workstation. After that, the CPLEX-D 12.6.2 solver was applied in order to benefit from the performance of the cluster. In particular, 8 machines of the cluster were harnessed for each computation. In this context, the three aforementioned settings (Concurrent Mode, Distributed Mode, Combined Mode) were tested consecutively in separate runs. As a result, the optimal objective value and the computation time were recorded. In accord with practical requirements of planning, the calculations were terminated automatically, if an optimal solution could not be found within 2 h. Then, the best feasible solution and the relative gap between this solution and the upper bound of the objective value were taken from the log file (see Table 7).

For the first scenario (No. 1), which was derived from the aforementioned numerical example illustrated in Fig. 5, the five calculations were carried out until optimality was reached, even if the time limit of 2 h was exceeded. As might be expected, the identical objective value of $ 63,512 resulted in all cases due to the equivalence of the linear and the nonlinear model formulation. However, while the CPLEX-D solver in the Concurrent Mode was able to solve the MILP within 15 s, the SCIP solver, applied to the MINLP, needed about 3 h and 45 min for the same result. Moreover, it was not possible for the SCIP solver to find proven optimal solutions in all the following 29 scenarios (No. 2–30) within the given time limit. Only feasible objective values were an outcome of the calculations in these scenarios. The conventional CPLEX solver applied to the model on a single machine of the cluster was able to find 21 optimal solutions with respect to the time limit. Even with regard to the maximum gap and the average gap, this solver dominates the CPLEX-D alternative (which makes use of 8 machines of the cluster in each computation), if the Distributed Mode or the Combined Mode is selected. However, it becomes obvious from the summarized statistical results of the scenario analysis (see Table 8) that applying the CPLEX-D solver in the Concurrent Mode led to the best overall performance. All of the 30 scenarios were solved to the optimum in not more than 1 h and 20 min.

The equivalent linearization of the proposed nonlinear programming model creates the prerequisite for using the benefits of the high-performance solvers CPLEX and CPLEX-D. The benefits of this software are additionally revealed by the fact that the best feasible solutions generated by the alternative SCIP solver within the time limit of 2 h are up to 31.47 % (on average 10.56 %) lower than the optima of the best performing CPLEX-D solver in the Concurrent Mode.

6 Conclusion

This article deals with the development of a mixed-integer nonlinear programming model for continuous-time production, distribution and financial planning with periodic liquidity balancing and its application to randomly generated data sets. We consider a multi stage SC network within a short-term planning horizon of several days. In this context, a novel hybrid form of continuous-time and discrete-time modeling is applied. This allows for the simultaneous planning and scheduling of production, transportation, sales and financial transactions exactly to the minute, as well as the periodic balancing of monetary consequences within liquidity periods. Thus, not only optimal quantities of production, transportation, sales and financial transactions, but also optimal durations, starting and ending times of these events can be determined. The scope of SC operations depends on the profitability of meeting a given level of demand for several goods produced in different SC stages, which is not necessarily satisfied in full. According to external requirements, time windows for meeting demand can be set. SC operations inherently yield revenue after completion. Thus, available bank loans with given terms and credit rates can be used for initiating these operations, if no internal financing is possible. Liquidity management includes the daily balancing of all monetary consequences within the system, taking profit maximization at the end of the planning horizon into account. Due to the high complexity of coordination between the financial and the operational level, it was shown even for our illustrative small-scale example (see Fig. 4) that only an equivalent linearized version of the model was solvable to optimality with a commercial optimization software package.

Although the numerical results are valid only for the case study, the conceptual model is easily adaptable to the data situation of other businesses. In general, managerial insights into the profitability of an entire SC network can be gained in a situation with a lack of initial capital (especially applicable to small- and medium-sized companies) and with respect to given demand, operating conditions and credit terms. In the case of profitability, time-specific instructions for each site in the network, and for the network managers, can be derived from the optimal solution.

Besides the production stages considered in our modeling, additional SC stages for storage and distribution can be added, if the production coefficient is set to zero. Investment alternatives can be implemented, if the algebraic signs of the monetary consequences of the financings are inverted. In this context, holding cash would be a special form of an investment with an interest rate of zero. Further research may focus on the distribution of profits among network partners and network managers, or on the coordination with longer-term planning, such as the relationship between structural investments and site usage or between short-term bank loans and long-term debt.

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